Calculus & Physics

Calculus is used a lot in the more advanced levels of physics. What it allows the physicist to do is examine motion (or some other physical phenomenon) during infinitesimal (aka teensy-weensy) times or distances. It's sort of like a mathematical microscope.

Newton "invented" calculus in order to better analyze the physics of his studies. For our purposes, we're going to take a brief look at the calculus involved in some fundamental physics concepts, such as *linear motion*, *simple harmonic motion*, and *work* (to name a few). To do this, we need two very basic calculus operations : the ***derivative*** and the ***integral***.

1) DERIVATIVES

A derivative is "simply" the *slope of the tangent to a curve* at a given point (aka the *instantaneous* rate of change). We did this last year, but we needed graphs and rulers. This year, all that is needed is the function, y = f(x), whose derivative is needed at a particular value for x. The derivative of y = f(x) is symbolized in one of 3 ways (all of which mean the same thing!) :

f'(t) **dy/dt** **y'**

Often, in physics, instead of using **x** we use **t** (for time). Therefore, y = f(t) and the derivative would be expressed as **f'(t)**, **dy/dt**, or, of course, **y'**. You dig?

Formula for the Derivative

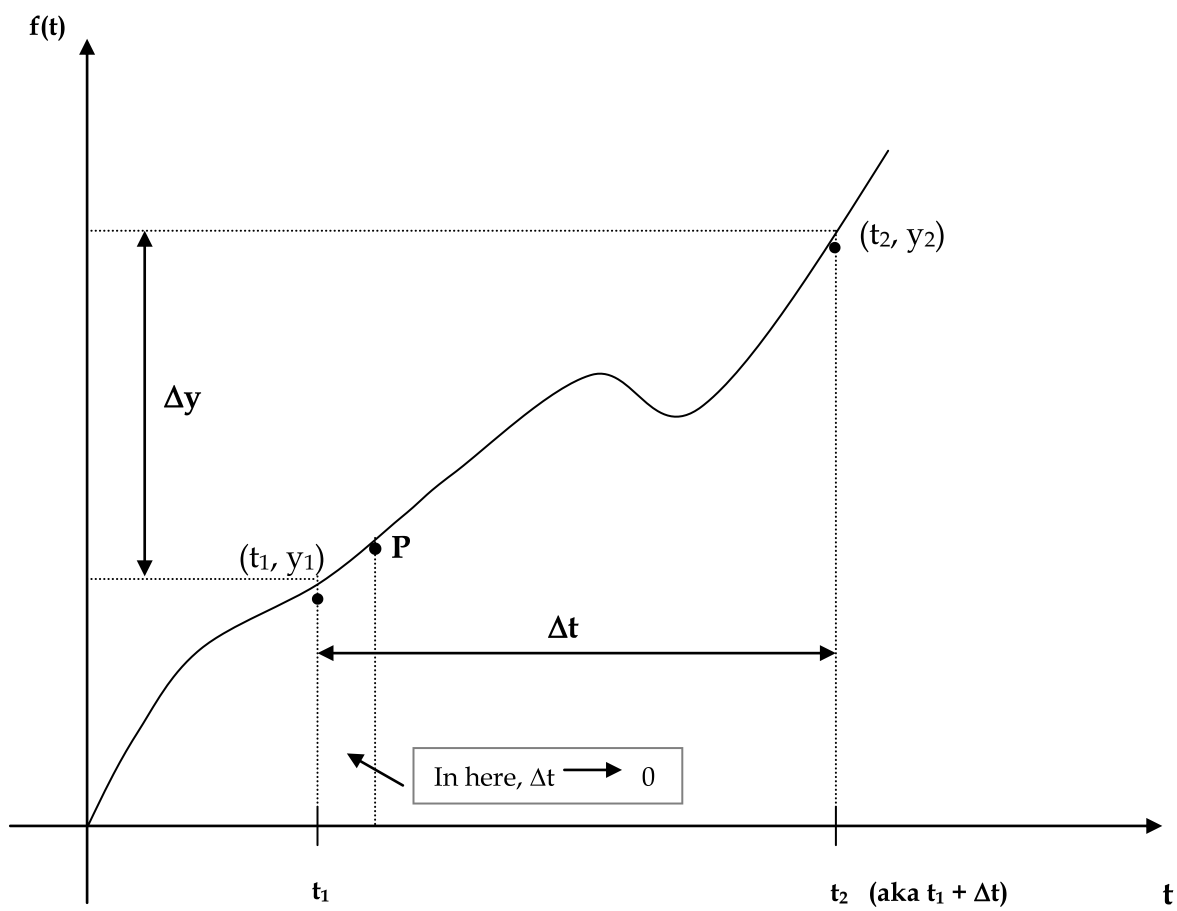
$$f'(t) = \lim_{\Delta t \to 0} \frac{[f(t + \Delta t) - f(t)]}{\Delta t}$$

Hey! This expression is just a fancy way of finding the slope Δy/Δt of a line where **Δy** has been replaced by **[f(t + Δt) - f(t)]** . The "$\lim_{\Delta t \to 0}$" means that you are limiting yourself to such a small segment of the curve that it's essentially straight during that interval. As a result, the slope of a line formula can then be used within that "teensy-weensy" section.

Reference : The Meaning of the Formula for f'(t)

On the graph below, we'd like to find the derivative (i.e., the *instantaneous* slope) of the curve at point (t_1, y_1). So, suppose that we pick *any* second point (t_2,y_2) that is on the curve. If we substitute the expressions (t_1+ Δt) for t_2, f(t_1) for y_1, and f(t_1+ Δt) for y_2, then the slope between these two points will look like our formula. The only problem is, as you can see, this slope does **not** look like the shape of the curve at point (t_1,y_1)!!!

The problem, of course, is that old devil, Δt. It's too large. If Δt was real small (teensy-weensy), then the line drawn between the two points would look like the shape of the curve for that interval. On our graph below, imagine that point (t_2, y_2) is chosen at point **P**. Then our formula would work because we've made sure that Δt is small, or as they say in the lingo of calculus, we've made sure that $\Delta t \rightarrow 0$.



Your Very First Derivative Rules

The theory behind the derivative is nice, and you'll learn a lot more about it in your calculus course in the fall. Right now, we're going to "cut to the chase": the rules for derivatives that we need to do physics *at the start of the school year*.

We're only going to do the first 3 basic rules, because that's all you'll need to start the AP Physics course in September. (Don't worry, you'll learn a lot more rules during the course of the school year.) These rules are based on proofs involving the "long form" of the derivative, found on the first page of these notes. You'll do these proofs in AP Calculus, but *you're not responsible for them in AP Physics*!!! Woo hoo!!!

For the expressions below, **k**, **c**, and **n** are *constants*; **u** and **v** are either *functions of time* or *functions of position.*

1. **d/dt (k) = 0**

   This rule says that the derivative of a constant is 0. Piece of cake!

   Example: If f(t) = 7, then f′(t) = 0

2. **d/dt (kt^n) = (kn)t^{n-1}** (This is known as *the power rule*)

   This rule says that if the variable (**t**) is raised to a power of **n**,
   then the derivative is found by multiplying (a) the coefficient (**k**) times
   (b) the power (**n**) times (c) the variable (**t**) *raised to* the power **n-1**.

   So, just identify the **n** value and the **k** value and plug into the rule.

   Examples: i. $f(t) = 3t^2$: here, **n** = 2 and **k** = 3 ∴ $f'(t) = 6t^1 = 6t$

   ii. $f(t) = (1/2)t^4$: here, **n** = 4 and **k** = 1/2 ∴ $f'(t) = 2t^3$

3. **d/dt (u + v) = u' + v'**

   This rule is easy. It just says that the derivative of a sum is simply the sum of the derivatives of the individual terms that make up the sum.

"The derivative of the sum is the sum of the derivatives"

Example: If $f(t) = 2t^3 - 4t + 6$, then $f'(t) = 6t^2 - 4$

Exercise 1

Use the "rules" to find the derivatives for the functions f(x) shown below.

1) $f(t) = 3t^2$

2) $f(t) = 2t + 3$

3) $f(t) = (1/4)t^3$

4) $f(t) = -(3/4)t^{1/2}$

5) $f(t) = t^2 + t + 1$

6) $f(t) = t^3 - 5t$

7) $f(t) = (1/2)t^2 + 6$

8) $f(t) = (2/3)t^4 - 4t + 2$

9) $f(t) = (1/5)t^3 + 3t$

10) $f(t) = 1/x + 1/x^2$ (Big hint: express this as $x^{-1} + x^{-2}$, and apply the power rule…)

Applications of the Derivative

The most common application of the derivative that we use in AP Physics is the "position – velocity - acceleration" application. (By the way, in AP Physics we use "**x**" to represent position, not "**d**", which is what we used this past year.)
Since **velocity** is ***the slope of a position vs time*** and **acceleration** is the ***slope of velocity versus time***, then we can say (in "calculus-speak") that *velocity is the derivative of position* and *acceleration is the derivative of velocity*.

There are a bunch of other uses for the derivative in AP Physics, but we'll get to them during the school year. So, for the time being,

Given: **x(t)** **position** as a function of **time**

Then: **dx/dt = velocity** **and** **dv/dt = acceleration**

> Here's a tip that may help you to remember whether or not to use a derivative:
>
> If a quantity can be expressed as a *quotient* in its "basic" form, then that same quantity can be found by taking a *derivative*.
>
> For example, **v = x/t** and **I = q/t** (basically)
>
> Then, in the world of calculus, **v = dx/dt** and **I = dq/dt**

Example 1

A particle's position is given by: $x = 4t^3 - 2t$, where x is in meters and t is in seconds.
(a) Find the velocity of the particle at time t = 3 seconds.
(b) Find the time at which the particle stops momentarily
(c) Find the particle's acceleration at time t = 6 seconds.

Solution:

$v = dx/dt = 12t^2 - 2$ and $a = dv/dt = 24t$

(a) $\therefore v_3 = 12(3)^2 - 2 = 106$ m/s

(b) if v = 0, then $0 = 12t^2 - 2$

$\therefore 2 = 12t^2$

$\therefore t^2 = 1/6$

$\therefore t = 0.41$ seconds

(c) $a_6 = 24(6) = 144$ m/s^2

Problem Set 1

1. The distance in meters that a particle can move in time t seconds is given by: $\mathbf{x = 180t - 5t^2}$. At what time does the particle stop moving?

2. A particle projected vertically reaches an elevation, in meters, given by: $\mathbf{y = 50t - 5t^2}$, where t is in seconds. How fast is the particle moving when it reaches an elevation of 75 meters?

3. A particle's position in meters is given by : $\mathbf{x = t^3 - 3t}$, where t is in seconds.
   (a) Find the particle's velocity at time 5 seconds.
   (b) Find the particle's acceleration at time 2 seconds.

4. A particle's position in meters is given by: $\mathbf{x = 2t^3 - 4t}$, where t is in seconds. Find the particle's acceleration at each instant for which the speed is zero.

5. A particle's position is given by: $\mathbf{x = 3t - t^3}$
Find the following:

(a) The *average* speed between t = 0 second and t = 2 seconds.
(**Note**: $v_{avg} = \Delta x/\Delta t$… *always*, for *any* motion)

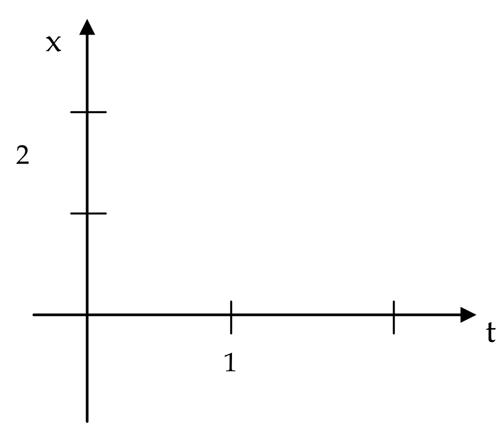
(b) The instantaneous *velocity* at t = 2 seconds

(c) The instantaneous *speed* at t = 3 seconds

(d) The *average* acceleration between t = 0 seconds and t = 2 seconds
(**Note:** $a_{avg} = \Delta v/\Delta t$… *always*, for *any* motion)

(e) The *instantaneous* acceleration at t = 2 seconds

(f) *Sketch* a graph of x vs t by doing the following:



- Find x when t = 0 (where the graph starts)

- Find t when x = 0 (where x crosses the axis)

- Find where the slope (the derivative!) = 0
(this tells you where the graph "turns around")

2) INTEGRALS

The integral has 2 interpretations:

- It represents **the function whose derivative is given**. In other words, *the integral is the inverse function of the derivative*. This type of integral is called the ***indefinite integral***. It is expressed by

$$y = f(t) = \int f'(t)\,dt$$

and it is found by using the rules of integration found on page 9.

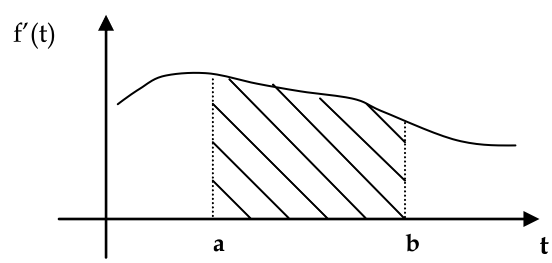
When using the rules, you'll notice that a "**c**" is always present . "c" is called the *constant of integration*. This value is found based on the *initial conditions*. In other words, **c** is the value for the function **y** when **t** = 0.
For our *immediate* purposes, the constant of integration will equal the initial velocity or the initial position.

- It can also represent the ***area under the curve of f'(t) vs t*** between t = a and t = b. This is known as a ***definite integral***. It will **always** yield a number answer! We did this, in effect, last year when we found the area under a ***v vs t*** graph in order to determine displacement.

  As you will learn more about in calculus, this type of integral represents an *infinite sum* of the areas of "skinny" rectangles (of length f'(t) and width Δt) between the values t = a and t = b. The "skinnier" these rectangles are, the more their sum matches the area under the curve. This will happen when $\Delta t \longrightarrow 0$.

The definite integral is expressed by

$$\int_a^b f'(t)\,dt = f(b) - f(a)$$



You **cannot** find the definite integral without first finding the indefinite integral. And you can't find the indefinite integral without using the rules!!!
By the way, *there is no need to "worry" about the constant of integration* for a definite integral because it subtracts itself away!!! Woo hoo!

Your Very First Integration Rules

1. $\int (k)dt = kt + c$

2. $\int (kt^n)dt = [k/(n+1)]t^{n+1} + c$

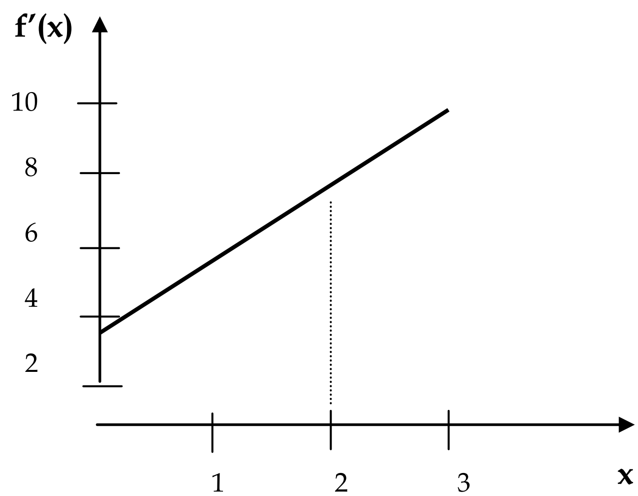
For Rule 2, **n** doesn't have to be a whole number…it can be 1/2, for example.

3. $\int (u + v)dt = \int (u)dt + \int (v)dt$

As with derivatives, you'll be getting more rules as the school year goes along. We just don't need them to start the school year.

Example 2 (for integrity's sake): Use the rules to determine the *definite integral* of the function f′(x) shown below. Then measure the corresponding *area* under the graph of f′(x) vs x for the given interval, and compare to the value of the definite integral.

$$\int_0^2 (2x + 3)\, dx$$



Using Rule 3, we get

$$\int(2x + 3)\, dx = \int (2x)\, dx + \int(3)\, dx$$

Using Rules 2 & 1, we get

$$f(x) = (x^2 + 3x)\Big|_0^2 = (10) - (0) = 10$$

∴ The area under the graph should = 10. Does it?

Exercise 2 : Determine the following ***indefinite*** integrals.

1) $\int (5t - 2)\, dt$

2) $\int t^4\, dt$

3) $\int (4t^3 + 3t)\, dt$

4) $\int (t^5 - t)\, dt$

Exercise 3 : Evaluate the following ***definite*** integrals.

1) $\int_1^2 (4t - 5)\, dt$

2) $\int_0^3 [6 - (1/2\sqrt{t})]\, dt$

3) $\int_1^4 [(1/2\, t^3) + 9t]\, dt$

Applications of the Integral

Below you'll find the first couple of integral applications that we'll need in AP Physics. As with derivatives, these are NOT the only applications. We'll get more as the school year progresses

1. Given: $a(t)$ (**acceleration** as a function of **time**)

   Then: $\int a\,dt$ = **velocity** and $\int v\,dt$ = **position**

> The only time (for now) that you'll have a "**c**" value is for **velocity** and **position**. For these 2 cases, the "c" value will be the initial velocity or the initial position. The applications shown below (for **work**, **energy**, and **impulse**), will *all* be *definite* integrals, so the "c" value is irrelevant! Woo hoo!

2. Given: $F(x)$ (**force** as a function of **position**)

   Then: $\int F(x)\,dx$ = **Work**

3. Given: $P(t)$ (**power** as a function of **time**)

   Then: $\int P\,dt$ = **Energy** (or Work)

4. Given : $F(t)$ (**force** as a function of time)

   Then: $\int F(t)\,dt$ = **Impulse** (J)

> Here's a tip that may help you to remember whether or not to use an integral:
>
> If a quantity can be expressed as a *product* in its "basic" form, then that same quantity can be found by *integration.*
>
> For example, $\mathbf{x = vt}$ (distance = rate x time, right?)
>
> Therefore, in the world of calculus: $\mathbf{x = \int v\,dt}$

Example 3

A particle has an acceleration given by **a = 2t – 1,** where **t** is in seconds and **a** is in m/s^2.

If the particle has an initial position of 17 m and an initial velocity of 5 m/s, find:

(a) the velocity of the particle as a function of time

$v = \int a \, dt = \int (2t - 1) \, dt$

$\therefore \quad v = t^2 - t + c_1$

Since the initial velocity is 5 m/s, then $c_1 = 5$

$\therefore \quad \boxed{v = t^2 - t + 5}$

(b) the position of the particle as a function of time

$x = \int v \, dt = \int (t^2 - t + 5) \, dt$

$\therefore \quad x = t^3/3 - t^2/2 + 5t + c_2$

Since the initial position is 17 m, then $c_2 = 17$

$\therefore \quad \boxed{x = t^3/3 - t^2/2 + 5t + 17}$

(c) the speed and position at t = 2 seconds

$v_2 = 2^2 - 2 + 5 = 7$ m/s

$x_2 = 2^3 - 2^2/2 + 5(2) + 17 = 33$ m

Example 4

The **force** acting on a block is given by $\mathbf{F = x^2 - 3x}$, where **x** is in meters and **F** is in Newtons. Find the **work** done on the block between x = 2 and x = 5 meters.

$$\text{Work} = \int_2^5 F(x)\,dx = \int_2^5 (\mathbf{x^2 - 3x})\,dx$$

> Remember! This is a definite integral! So there's no need to "worry" about the constant of integration!

$$= \left[x^3/3 - 3x^2/2\right]\Big|_2^5$$

$$= [\mathbf{5}^3/3 - 3(\mathbf{5})^2/2] - [\mathbf{2}^3/3 - 3(\mathbf{2})^2/2]$$

$$= [4.17] - [-1.33]$$

$$= 5.5 \text{ J}$$

Problem Set 2

1. The **acceleration** due to gravity here on earth is **-9.8 m/s^2**. If a ball is launched straight up in the air with an initial speed of 30 m/s, and an initial height of 2 meters. Find its velocity and position after the first 4 seconds of the flight.

2. A car moving at 17 m/s **decelerates** at a *constant* rate of **3 m/s^2**. Find:
(a) how long it takes the car to stop
(b) the car's displacement during that time (assume an initial displacement of 0 m).

3. A particle is accelerated at a rate given by : **a = 3 + 2t**, where t is time in seconds. Initially, the particle has a speed of 4 m/s. Find:
(a) the velocity as a function of time
(b) the displacement between t = 0 and t = 4 seconds.

4. The net force (in Newtons) acting on a mass is given as a function of position (in meters) by: $\mathbf{F = -7x}$

   Find the **work** (in Joules) done in moving the mass from x = 1 meter to x = 3 meters.

5. A machine uses power (in Watts) at a rate given by: $\mathbf{P = 4t - (1/3)t^2}$, where t is in seconds.

   Find the **energy** used by the machine between t = 2 seconds and t = 4 seconds.

6 A force (in Newtons) applied to a mass is given as a function of time (in seconds) by:

$$\mathbf{F = 3t^2 - 2}$$

Find the **impulse** (in N•sec) delivered by the force to the mass from t = 1 second to t = 5 seconds.

Answers

Exercise 1

1. $6t$
2. 2
3. $(3/4)\ t^2$
4. $(-3/8)t^{-1/2}$
5. $2t + 1$
6. $3t^2 - 5$
7. t
8. $(8/3)t^3 - 4$
9. $(3/5)t^2 + 3$
10. $-x^{-2} - 2x^{-3}$

Problem Set 1

1. 18 sec
2. (+/-) 32 m/s
3. 72 m/s, 12 m/s^2
4. 9.75 m/s^2
5. –1 m/s, -9 m/s, 24 m/s, -6 m/s^2, -12 m/s^2, you graph it, baby!

Exercise 2

1. $(5/2)t^2 - 2t + c$
2. $t^5/5 + c$
3. $t^4 + (3/2)t^2 + c$
4. $t^6/6 - t^2/2 + c$

Exercise 3

1. 1
2. 16.3
3. ~ 99

Problem Set 2

1. ~ (-10)m/s, 43.6 m
2. 5.66 sec, 48.17 m
3. $t^2 + 3t + 4$, 61.3 m
4. – 28 J
5. 17.76 J
6. 116 N•s